

Scientific Communication

## ROBUST MULTI-OBJECTIVE STATIC OUTPUT FEEDBACK CONTROL BASED ON $H_2/H_\infty/\mu$ COMBINATION

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### ABSTRACT

This paper presents an overview on an output feedback controller with a combination of  $H_2/H_\infty/\mu$ . The design objective is a mixture of robust stability, nominal/robust performance, strict limitations on control signal and minimization of disturbance effects. In a physical system, the several targets contribute in a system control. Each one of the nominal and robust performance targets has their own strengths and weaknesses. A new approach in the presented paper is a combination of the two output feedback controllers of  $\mu$  and  $H_2/H_\infty$ . When all objectives are formulated in terms of a bounded real lemma, controller design results in a solution for a system of LMI. The purpose of the presented paper is to make balance between the nominal and the robust performance of output feedback. First, we use mixed  $H_2$  and  $H_\infty$  norm for a nominal performance target while the other use  $\mu$  synthesis for the robust performance. By combining these two controllers, the procedure of weights achievement will be formulated. Finally, modeling of an unmanned aircraft is applied to show the effectiveness and benefits of this method.

**Keywords:**  $H_2/H_\infty/\mu$  controller, LMI, multi-objective output feedback, uncertain dynamic systems, single person aircraft.

### INTRODUCTION

Unmodeled dynamics, non linearity of systems and the availability of disturbance are among some of the reasons explaining why the linear control systems theory has never reached to the ideal solution. For this reasons, several targets have been employed in a system control (Mashayekhi *et al.*, 2013). Robust Stability means that the system will be stable with uncertainty, while the nominal performance which implies considering the system operation without uncertainty, has decisive effect on the operation of a system. By robust performance, we mean considering the system operation with uncertainty. It is obvious that whenever the singular values of controller are higher, the performance of systems will be more desirable, but at the same time, it provides higher chances of saturation occurrence. In order to consider the robust performance, we used  $\mu$  analysis. Operating limitation on controlling signal increase of controlling signal leads to saturation of the actuators.  $H_2$  norm essence can be responsible for such targets. Minimizing disturbance effect distortion can result in undesirable effect of transient response, therefore, reduction of the effect of disturbance, is one of the controlling targets. Mixed norm of  $H_2$  and  $H_\infty$  can be a useful strategy to reach the mentioned controlling targets. To date, several studies

have been performed on the mixed norm and the multi-objective control. This paper tends to reduce controlling signal, robust performance and stability and design weight functions. One of the new approaches of this paper is the combination of two controllers of  $\mu$  and  $H_2/H_\infty$  based on output feedback. The controller for robust stability status, nominal performance, robust performance and noise reduction are continuously designed. First, the controller of  $H_2/H_\infty$  will be designed for nominal performance targets, robust stability and noise reduction, and then  $\mu$  controller will be designed for robust performance. Now, add up these two controllers and achieve their weights with LMI. Controller will be achieved from solving the optimization problem. At first, a controlling problem will be changed to LFT standard form, considering uncertainty, then, the status equations will be written and by use of the constraint's weight function the robust controlling targets will be reached. The static output-feedback problem is one of the problems in systems and control theory that has been researched a lot. The use of output feedback provides the flexibility and simplicity of implementation. Moreover, in practical applications, full state measurements are not usually possible. Therefore, the restricted-measurement static output-feedback problem is of the issues with extreme importance in practical controller design

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applications such as flight control. The first formulation of the  $H_\infty$  control problem was performed in 1981 by Zames. Next to Zames (1981) and Doyle *et al.* (1989, 1991) were the pioneers of robust control. To date, large numbers of researches have been performed to study of the robust control, the  $H_2$  control and  $H_\infty$  control. (Doyle *et al.*, 1989) analyzed the state space with  $H_\infty$  and  $H_2$  standard form and its solving. The conditions of solving problem and its solution using Hamiltonian matrix introduction could be mentioned as the highlights of this paper. Also Doyle *et al.* (1991) presented a tutorial overview on the linear fractional transformations (LFTs) and the roles of the structured singular value,  $\mu$ , and linear matrix inequalities (LMIs) in solution of LFT problems. Lescher *et al.* (2006) designed a multivariable, multi-objective controller to set the wind turbine. Controlling problem of this paper was the minimization of  $H_2/H_\infty$ . This problem was solved by LMI. His controller resulted in reduction of costs and mechanical depreciation. Also, it increased the lifelong of the system. Rotea *et al.* (1991) combined  $H_2/H_\infty$ , in this way, two important approaches were presented, 1)  $H_2$  optimized control with  $H_\infty$  bound (in fact a bounded optimization), and 2) Simultaneous  $H_2/H_\infty$  optimized control. In each step, the problem formulation and the controller were implemented. Scherer *et al.* (1997) presented an overview on the approach of linear matrix inequality (LMI) for the multi-objective synthesis of the linear output-feedback controllers. The design objectives could be a mixture of  $H_\infty$  performance,  $H_2$  performance, passivity, asymptotic disturbance rejection, time-domain constraints, and constraints on the closed-loop pole location. In addition, these objectives could be specified on different channels of the closed-loop system. In the work of Echchatbi *et al.* (2009) the robust static output feedback stabilization of an induction machine was addressed. The machine was described by a non homogenous bilinear model with structural uncertainties, and the feedback gain was computed via an iterative LMI (ILMI) algorithm. Pereira *et al.* (2004) addressed the mixed  $H_2/H_\infty$  robust control problem. An algorithm based on GAs and LMIs was proposed in order to find a fixed structure output feedback robust controller.  $H_\infty$  design has been considered for the static output feedback, Holl *et al.* (2004) addressed the applicability of the matrix-valued sum-of-squares (SOS) techniques for the computations of LMI lower bounds. In a study conducted by Gadewadikar *et al.* (2009) the problem of stabilization of an autonomous rotorcraft platform in a hover configuration exposed to external disturbances was discussed. Necessary and sufficient conditions were presented for the static output-feedback control of linear time invariant systems using the  $H_\infty$  approach. Prempain *et al.* (2001) claimed that the existence of a static output feedback control law is given in terms of the solvability of two coupled Lyapunov inequalities which results in a non-linear optimization

problem. However, by the use of state-coordinate and congruence transformations and by imposing a block-diagonal structure on the Lyapunov matrix to find the solution of a system of Linear Matrix Inequalities, they saw a reduction in the determination of a static output feedback gain, for a specific class of plants. Kanev (2004) expressed the reason of why the output feedback problem in the presence of uncertainty is a bilinear matrix inequality (BMI) problem, and BMI problems are not convex. Actually, such problems have been shown to be NP-hard which means that they cannot be expected to have polynomial time complexity. Raissi Dehkordi *et al.* (2009) dealt with the robust performance problem in a linear time-invariant control system in the presence of the robust controller uncertainty. Assuming that the plant uncertainty is modeled as an additive perturbation; a geometrical approach was followed in order to find a necessary and sufficient condition for the robust performance in the form of a bound on the magnitude of controller uncertainty. This method is performable for the SISO systems. Authors know that this method is more efficient than the approach of structured singular value. Another study, Mashayekhi *et al.* (2013) presented a state feedback control of linear time invariant systems using the  $H_2/H_\infty/\mu$  approach. The rest of this paper is organized as follows: Section I establishes the problem which will be addressed and the  $H_2/H_\infty$ ,  $\mu$  and  $H_2/H_\infty/\mu$  combination control will be demonstrated. Section II presents the example design of Single Person Aircraft (X-29). In Section III, the approach will be illustrated and the results of the simulations will be discussed.

## Problem Statement

### A. $H_2/H_\infty$ Controller

The existence of uncertainty is due to an uncertain and erratic input (for example noise and disturbance) and the Unmodeled dynamic is caused when we cannot completely and precisely describe a true system by a mathematical model at all. On the other hand, the important issues of a true system are the following objects: robust stability, robust and nominal performance, settling time, and maximum over shoot and so on, which try to gain these objectives about the controlling problem (Akbar *et al.*, 2009; Lescher *et al.*, 2006; Rotea *et al.*, 1991). The type of uncertainty is an important problem in analysis. Zhou *et al.* (1994) and Liu (2002) researched on the optimization approach of mixed  $H_2$  and  $H_\infty$  norm.

According to small gain Theorem, a system shown in figure 1 is well-posed and internally stable for all

$$\Delta(s) \in RH_\infty \text{ with } \|\Delta\|_\infty < \gamma^{-1} \text{ if and only if } \|M\|_\infty < \gamma.$$

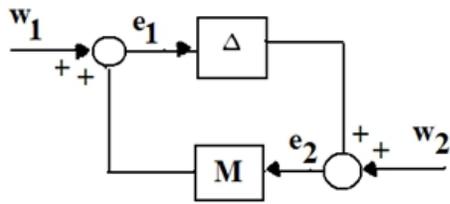


Fig. 1. M – Δ MODEL.

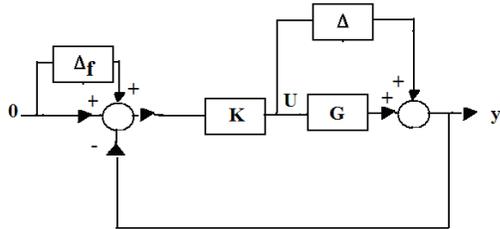


Fig. 2. Additive uncertainty.

Additive uncertainty shown in fig. 2 robust stability task is:  $q = (I + KG)^{-1} K P \Rightarrow \|(I + KG)^{-1} K\|_{H_\infty} < \gamma^{-1} (1)$ .

The objective for the inner loop control is to design an output feedback law such that the close loop system satisfies the following performance specifications:

**Objective 1:** if  $\Delta = 0$  then  $\|FS\|_\infty < 1$  (nominal performance).  $S = (I + GK)^{-1}$  ( $S$  is sensitivity function and  $F(s)$  is weighting function).

**Objective 2:** if  $\Delta \neq 0$  then system has been robust stability.  $M = (I + KG)^{-1} K$ , if  $\bar{\sigma}(\Delta(j\omega)) \leq \gamma(j\omega) \Rightarrow \|\gamma(S)M\|_\infty < 1$

**Objective 3:**  $n$  is white noise with one PSD (power spectral density).  $H_2$  Norm, cause decreasing of controlling signal.  $\|T_{nU_1}\|_{H_2} < 1$  (To minimize  $U_1$  variance with noise input). (Mashayekhi *et al.*, 2013)

Then we have three tasks for controller design ( $\|FS\|_\infty < 1, \|\gamma(S)M\|_\infty < 1, \|T_{nU_1}\|_\infty < 1$ ), such that,

$$\begin{bmatrix} FS(K,G) \\ \gamma M(K,G) \\ RT((K,G)) \end{bmatrix} < 1 (2). \text{ Problem (2) shown in figure 4. Rotea}$$

and Doyle offer two methods to solve this problem (Rotea *et al.*, 1991; Zhou *et al.*, 1994). Mashayekhi *et al.* (2013) shows similar method for solution of the simultaneous multivariable controller. A large class of systems with uncertainty can be treated as LFT (Linear fractional Transformation). LFT model is shown in figure 3 (Zhou *et al.*, 1998).

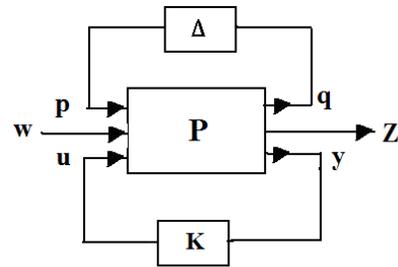


Fig. 3. LFT Model.

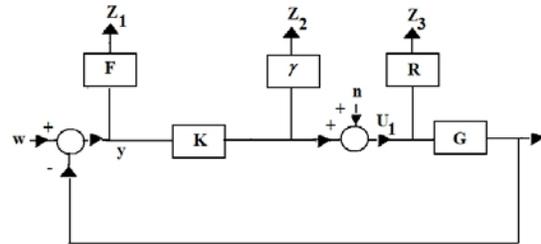


Fig. 4. Graphical model of problem (2).

$W$ : the disturbance signals to the system which will not be a function of states of the system,  $Z$ : the variable that will be controlled,  $P$ : the nominal open loop system,  $Y$ : the system measurable output. To transform the changed diagram of figure 4 to the LFT model, we will be written the problem to standard form, and we will be dissolving using of riccati equation (Scherer, 1990). The (2) LFT model is practicable in form (4) and it can be used to design a controller by theorem 2. The state space of figure 4 is written in (4).

Determining 3 weight matrices, specified in figure 4, contain special importance. Using robust optimal output feedback method for (4) equations, and this is a new approach.

$$\begin{aligned} \dot{x} &= Ax + B_1 W + B_2 u \\ z &= C_1 x + D_{11} W + D_{12} u \\ y &= C_2 x + D_{21} W + D_{22} u \end{aligned} \quad , \quad Z = \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} FS \\ \gamma M \\ RT \end{bmatrix} r \quad (3)$$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_f \\ \dot{x}_\gamma \\ \dot{x}_R \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 & 0 & 0 \\ -B_f C & A_f & 0 & 0 \\ 0 & 0 & A_\gamma & 0 \\ 0 & 0 & 0 & A_R \end{bmatrix}}_{A_{CL}} \begin{bmatrix} x \\ x_f \\ x_\gamma \\ x_R \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & B & B \\ B_f & -B_f D & -B_f D \\ 0 & 0 & B_\gamma \\ 0 & B_R & B_R \end{bmatrix}}_{\begin{bmatrix} B_1 & B_2 \end{bmatrix}} \begin{bmatrix} r \\ n \\ u \end{bmatrix}$$

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -D_f C & C_f & 0 & 0 \\ 0 & 0 & C_\gamma & 0 \\ 0 & 0 & 0 & C_R \\ -C & 0 & 0 & 0 \end{bmatrix}}_{\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}} \begin{bmatrix} x \\ x_f \\ x_\gamma \\ x_R \end{bmatrix} + \underbrace{\begin{bmatrix} D_f & -D_f D & -D_f D \\ 0 & 0 & D_\gamma \\ 0 & D_R & D_R \\ 1 & -D & -D \end{bmatrix}}_{D_{CL}} \begin{bmatrix} r \\ n \\ u \end{bmatrix} \quad (4)$$

*B. μ Controller*

Here we try to assess robust performance of this closed-loop system using  $\mu$ -analysis associated. Robust performance condition is equivalent to the following structured singular value  $\mu$  test (Doyle *et al.*, 1991).

$$\|T_{WZ}(M, \Delta)\|_{\infty} < \gamma^{-1}, \forall \|\Delta\|_{\infty} < \gamma \Leftrightarrow \mu_{\Delta P}(M) < \gamma, \forall W \quad (5)$$

The complex structured singular value  $\mu_{\Delta}(M)$  is defined

$$\text{as } \mu_{\Delta}(M) = \frac{1}{\min\{\bar{\sigma}(\Delta) | \det(I - M\Delta) = 0\}}$$

Lower and Upper bound of  $\mu$  can be shown to  $P(UM) \leq \mu_{\Delta}(M) < \min \bar{\sigma}(DMD^{-1})$  (Packard *et al.*, 1993).

1) *D-K iteration*

Unfortunately, it is not known how to obtain a controller achieving the structured singular value test directly but, we can obtain the lower and upper bounds of  $\mu$ . Our approach taken here is the so-called *D-K iteration* procedure (Doyle *et al.*, 1992). First, for  $D = I$  fixed, the controller  $K$  is synthesized using the well-known state-space  $H_{\infty}$  optimization method. LFT form of figure 2 is written in equation 6.

$$\dot{x} = Ax + \begin{bmatrix} 0 & 0 & B \end{bmatrix} \begin{bmatrix} P \\ W \\ U \end{bmatrix} \quad (6)$$

$$\begin{bmatrix} q \\ z \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -C \\ -C \end{bmatrix} x + \begin{bmatrix} 0 & 0 & I \\ -I & I & -D \\ -I & I & -D \end{bmatrix} \begin{bmatrix} P \\ W \\ U \end{bmatrix}$$

C. *New approach:  $H_2/H_{\infty}, \mu$  combination*

Now, we tend to synthesize two controllers according to figure 5. As mentioned before, Nominal performance means considering system operation without uncertainty has decisive effect on the operation of a system. Robust performance means considering operation with uncertainty. It is obvious that whenever the singular values of controller are higher, systems performance is more desirable, but also it provides higher chances of saturation occurrences. So, we tend to balance between robust and nominal performances.  $W_1$  and  $W_2$  are weight functions as matrices in multivariable systems. Of course, it is important that robust performance contains nominal performance, so, controller coefficient of  $\mu$  should be smaller than  $H_2/H_{\infty}$  controller coefficient (Mashayekhi *et al.*, 2013). In a more explicit description, controller includes two parts, the first one using mixed  $H_2$  and  $H_{\infty}$  norm and the other using  $\mu$  synthesis. These two parts, include weights each of which have Important roles in systems control, because robust and nominal performance targets, has its own definiteness which their combination can create a new solution.

**Problem A:** Determine  $W_1$  and  $W_2$ , in a way that an additive uncertainty system contains robust stability.

$$M = (W_1K_1G + W_2K_2G + I)^{-1}(W_1K_1 + W_2K_2) \quad (7)$$

$$\|M\|_{\infty} < 1$$

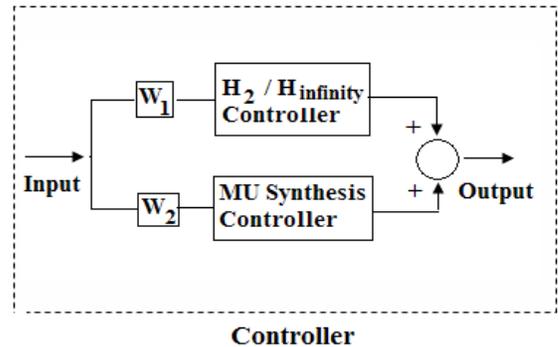


Fig. 5. Controller  $H_2/H_{\infty}/\mu$ .

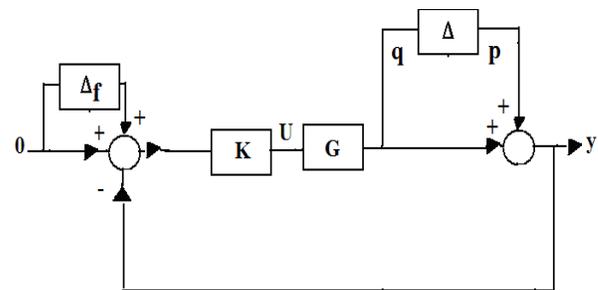


Fig. 6 Multiplication uncertainty.

**Problem B:** Determine  $W_1$  and  $W_2$ , in a way that a system having Multiplication uncertainty contains robust stability.

$$M = (GW_1K_1 + GW_2K_2 + I)^{-1}(GW_1K_1 + GW_2K_2) \quad (8)$$

$$\|M\|_{\infty} < 1$$

According to figures 2 and 6 we use state space to solve the problem A and B.

1) *Robust optimal static output feedback*

In this section we intended to follow the analysis of the conditioning of the pole placement problem with the multi-input case which is called the generalized output feedback. The Static Output-feedback (SOF) synthesis problem deals with a given class of systems, to derive theoretical conditions for the existence of a static control law and associate them with the numerical methods. The class of continuous-time, Linear Time-Invariant (LTI) systems is addressed. The systems are given as multi-input/multi-output state-space models. In addition to the control input vector and the measure output vector that define the control loop, the models may include some other input/output signals. They are introduced for input/output performance specifications defined by  $H_2$  and  $H_{\infty}$  norms and  $\mu$  synthesis. In this paper, we assume that the state of the generalized plant  $G$  is available for

the feedback. To be more precise, let a state-space description of P (Fig. 3) be given by (LFT Model):

$$\begin{cases} \dot{x} = A_{cl}x + B_{cl}w \\ y = C_{cl}x + D_{cl}w \end{cases} \quad (9)$$

$$A_{cl} = A + BKC \quad , \quad B_{cl} = B_w + BKD_w$$

$$C_{cl} = C \quad , \quad D_{cl} = D_w$$

The signal W refers to disturbance. The signals U and Y represent the control input and the measured output, respectively. After gaining  $K_1$  by  $H_2/H_\infty$  and  $K_2$  by  $\mu$  analysis, we tend to determine the weight functions, by the use of linear matrix inequality.

**Lemma1:** (bounded-real lemma) given a constant  $\gamma > 0$ , for system,  $M(s) = (A, B, C)$  the following two statements are equivalent, 1) this system is stable  $\| M(s) \|_\infty < \gamma$ , 2) A symmetric positive definite matrix Q exists such that: (Boyd *et al.*, 1994)

$$\begin{bmatrix} A^T Q + QA & QB_p & C_q^T \\ B_p^T Q & -\gamma^{-1}I & D_q^T \\ C_q & D_q & -\gamma^{-1}I \end{bmatrix} < 0 \quad (10)$$

$$Q > 0$$

**Lemma 2:** Consider the feedback system of Figure.3, where G is given by (8). Then, a given controller K is admissible and closed loop system is robust stability and desired performance if and only if there exists  $W_1$  and  $W_2$  solving the following LMI problem:

LMI of system (9), considering BRL theorem will be (11). ( $\beta = \gamma^{-1}$ )

$$\begin{bmatrix} A\Omega + BY_1C + BY_2C + C^T Y_1^T B + C^T Y_2^T B + \Omega A^T & B_w + BKD_w & \Omega C^T \\ (B_w + BKD_w)^T & -\beta I & D_w^T \\ C\Omega & D_w & -\beta I \end{bmatrix} < 0 \quad (11)$$

$$\Omega > 0 \quad , \quad Y_1 > 0 \quad , \quad Y_2 > 0$$

Where,  $Y_1 C = W_1 K_1 C \Omega, Y_2 C = W_2 K_2 C \Omega, \Omega = Q^{-1}, Q > 0 \Rightarrow \Omega > 0$

Proof: LMI of system (9), considering BRL theorem will be (12). ( $\beta = \gamma^{-1}$ )

$$\begin{bmatrix} (A + BW_1 K_1 C + BW_2 K_2 C)^T Q + Q(A + BW_1 K_1 C + BW_2 K_2 C) & Q(B_w + BKD_w) & C^T \\ (B_w + BKD_w)^T Q & -\beta I & D_w^T \\ C & D_w & -\beta I \end{bmatrix} < 0 \quad (12)$$

$$Q > 0 \quad , \quad W_1 > 0 \quad , \quad W_2 > 0$$

$$\begin{bmatrix} Q^{-1}(A + BW_1 K_1 C + BW_2 K_2 C)^T + (A + BW_1 K_1 C + BW_2 K_2 C)Q^{-1} & B_w + BKD_w & C^T \\ (B_w + BKD_w)^T Q & -\beta I & D_w^T \\ CQ^{-1} & D_w & -\beta I \end{bmatrix} < 0$$

$$Q > 0 \quad , \quad W_1 > 0 \quad , \quad W_2 > 0$$

We multiply the  $\begin{bmatrix} Q^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$  on the left and the right

of the matrix, define

$$Y_1 C = W_1 K_1 C \Omega, Y_2 C = W_2 K_2 C \Omega, \Omega = Q^{-1}, Q > 0 \Rightarrow \Omega > 0$$

Substituting into (12) yields:

$$\begin{bmatrix} A\Omega + BY_1C + BY_2C + C^T Y_1^T B + C^T Y_2^T B + \Omega A^T & B_w + BKD_w & \Omega C^T \\ (B_w + BKD_w)^T & -\beta I & D_w^T \\ C\Omega & D_w & -\beta I \end{bmatrix} < 0$$

$$\Omega > 0 \quad , \quad Y_1 > 0 \quad , \quad Y_2 > 0$$

In this method must be  $m=l$ . m, number of inputs and n, number of outputs.

## 2) METHODOLOGY

a. To design the  $H_2 / H_\infty$  output controller for the process with uncertainty. (it helps to select the weighting function properly).

b. For  $H_2/H_\infty$  design we can use Rotea and Doyle method. (Rotea *et al.*, 1991; Zhou *et al.*, 1994; Doyle *et al.*, 1994)

$$\text{or use } \left\| \begin{bmatrix} FS(K, G) \\ \gamma M(K, G) \\ RT((K, G)) \end{bmatrix} \right\|_\infty < 1 \text{ and obtained } K_1.$$

c. For  $F, \gamma$  and  $R$ , we use weighting functions to limit the magnitude of the sensitivity and complementary sensitivity functions.

d. To design the  $\mu$  output controller for the process with uncertainty (if the process is unstable, at the beginning, it must be stabilized). D-K iteration method can be used to improve the performance of the controller design for the system. Peak value of the  $\mu$  (D-K iteration) bound should be less than one, and obtained  $K_2$ .

e. Order reduction method can be used to reduce the order of the I+GK, and transfer to state space equation and given  $A_c$ .

f.  $K_1 = B^{-1} \times (A_{c1} - A)C^{-1}, K_2 = B^{-1} \times (A_{c2} - A)C^{-1}$ . A, B are in P-K system (LFT Model).

g.  $W_1, W_2$  are given with LMI (11) then the robust stability of the system has to be established.

h. H infinity norm of  $W_2$  must be smaller than  $W_1$ .

i.  $K = W_1 K_1 + W_2 K_2$ . This controller (K) has robust stability and desired performance.

## I. Example Design

### A. Single Person Aircraft(X-29)

In an airplane five main sections could be listed they are: motor, body section, landing system and wheels, wing and tail. The pitch angle of an airplane is controlled by adjusting the angle (and as a result, the lift force) of the rear elevator. The aerodynamic forces (lift and drag) as well as the airplane's inertia are taken into account. The X-29 aircraft is a recent example of a control configured vehicle which was designed with a high degree of longitudinal static instability (up to 35 percent at low subsonic speeds). The vehicle is stabilized by a full-authority, fly – by – wire flight control system. Prior to the flight, the linear models were used extensively to determine the close loop stability, controllability, and handling qualities with the various control system modes through the flight envelope. This section describes the commercial aircraft models which are now implemented. In the work of Bosworth (1992) which is a comprehensive report of NASA; the research has been conducted over X-29 state equation and model. Tae (2000) only designed the  $H_{\infty}$  controller over X-29. While Minisci *et al.* (2008) designed the multi-objective robust control for F-16 and F-18 airplanes. The X-29 airplane is a relatively small, single seat, high-performance aircraft powered by a single F404-GE-400 engine (General Electric, Lynn, Massachusetts). Its empty weight is 6350kg and the aircraft dimensions are shown in figure 7 also, the aircraft picture is shown in figure 8 in order to provide a low-drag configuration the vehicle incorporates a forward-swept wing with close-coupled canards. The airplane physical characteristics are listed in table 1.

The aircraft model is obtained by linearization of the nonlinear equations of motion about a 280 ft/sec (307km/h) landing configuration (Bosworth, 1992). The three input three output model which describes the longitudinal dynamics is given as follows (Bosworth, 1992; Tae, 2000):

Table. 1. X-29 physical characteristics.

6350 kg	weight	N.M	Maximum thrust force
8130			
3.437 m <sup>2</sup>	Canard area	$\alpha$	Angel attack
8.29m	Wing span	17.185 m <sup>2</sup>	Wing area
$\delta_{sf}$	Symmetric flap position	$\delta_{stf}$	Strake flap position
$\theta$	Pitch Euler angel	$\delta_c$	Canard position
v	Horizontal speed	$\dot{\theta}$	Pitch rate

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x = \begin{bmatrix} v & - & (\text{ft / sec}) \\ \alpha & - & (\text{rad}) \\ \dot{\theta} & - & (\text{rad / sec}) \\ \theta & - & (\text{rad}) \end{bmatrix} \quad (13)$$

$$u = \begin{bmatrix} \delta_c & - & (\text{deg}) \\ \delta_{sf} & - & (\text{deg}) \\ \delta_{stf} & - & (\text{deg}) \end{bmatrix}, \quad y = \begin{bmatrix} \theta & - & (\text{rad}) \\ v & - & (\text{ft / sec}) \\ \alpha & - & (\text{rad}) \end{bmatrix}$$

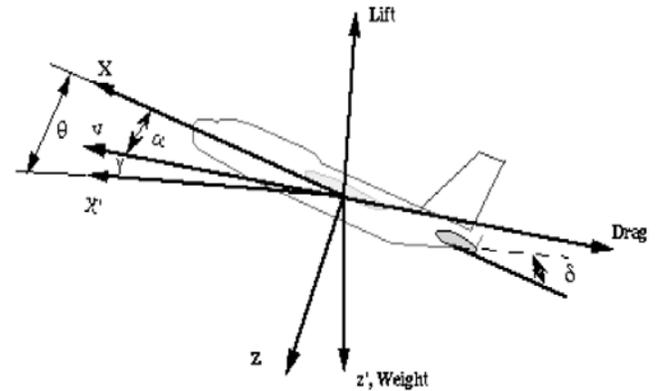


Fig. 7. Axes coordinate in aircraft.

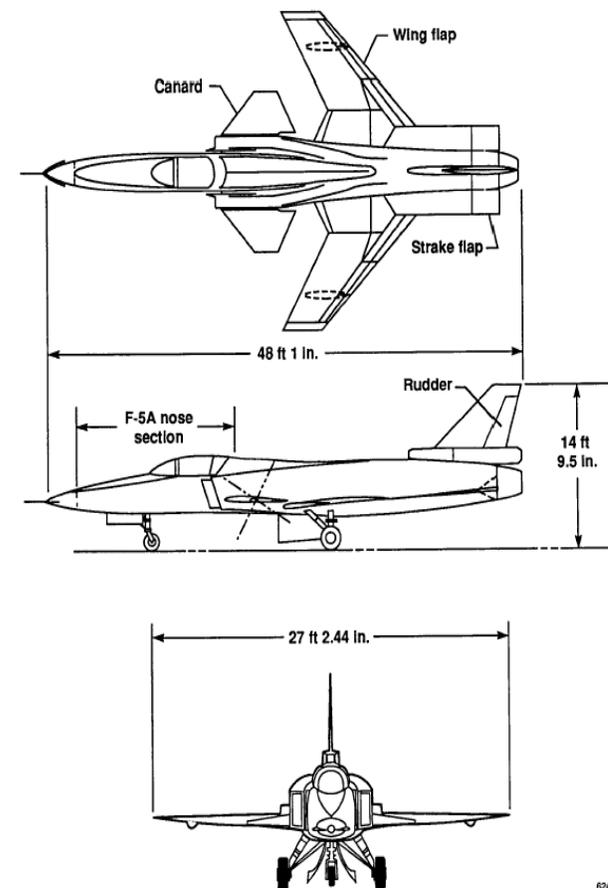


Fig. 8. X-29 airplane.

### III. Simulation results

The longitudinal dynamics of an aircraft has one natural mode: the short period mode. For the X-29, the short period mode, however, is composed of the stable mode and an unstable one. At first, we design  $H_2/H_\infty$  controller and design  $\mu$  controller. These controllers designed for P-K system. This system is four input four outputs. Then, for reducing the order of  $I + GK$ , a residualization method was implemented. By consideration of the practical

experiments and in accordance with equation (2), the weight functions selection with  $R = \frac{0.0001(s+0.01)}{s+1}I$ ,

$$F = \frac{0.2222s + 0.6667}{0.11s + 1}I, \quad \gamma = \frac{0.0001(0.25s + 2)}{1.5s + 0.0001}I \quad (\text{Lanzon, 2000}).$$

$K_1$  and  $K_2$  design with equations 2 and 8.  $W_1, W_2$ , are obtained via equation 11. According to figure 5,  $K$  is design. At first we designed the weight functions which

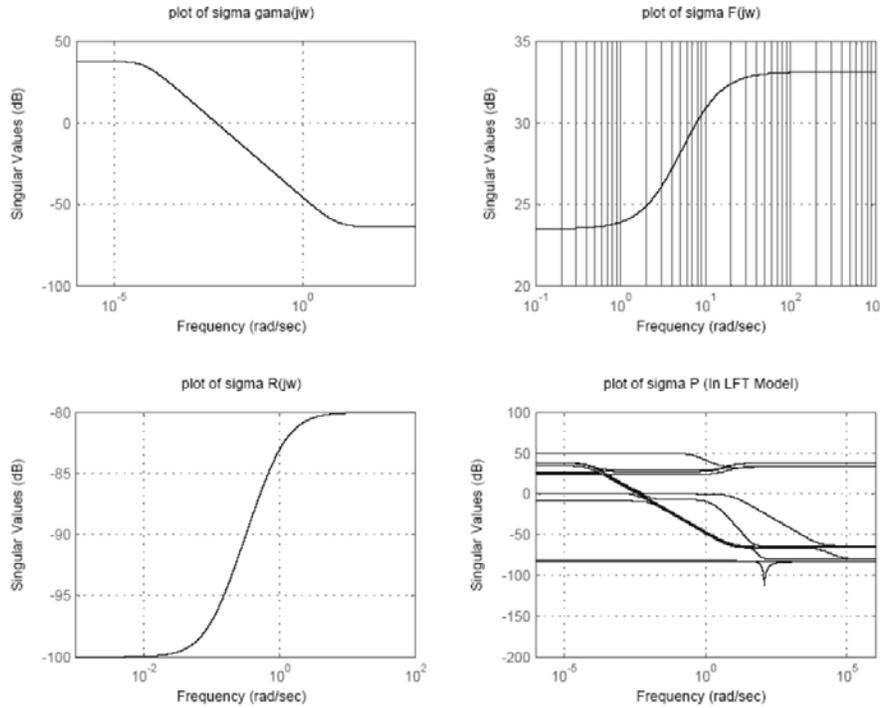


Fig. 9. Singular value for weighting functions and P (LFTModel).

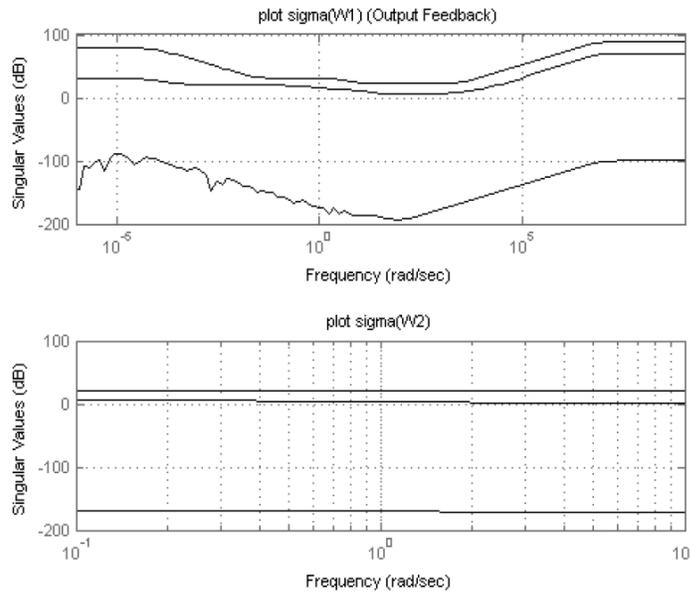


Fig. 10. Singular value for  $W_1$  and  $W_2$  Weighting function of  $H_2/H_\infty, \mu$  combination.

were drawn in figure 9, a, b, c. are selected the weighting functions (Beaven *et al.*, 1996; Sararth, 2011) by taking into account the practical experiments, The singular values of LFT model are drawn in figure 9-d. The Singular value for  $W_1$  and  $W_2$  Weighting function of

$H_2/H_\infty, \mu$  combination is shown in figure 10. While, the Singular value for T complementary sensitivity function is drawn in fig.11. the step response of T function is depicted in fig.12. It must be noted that, as it was mentioned, the system is multi input-output; and the

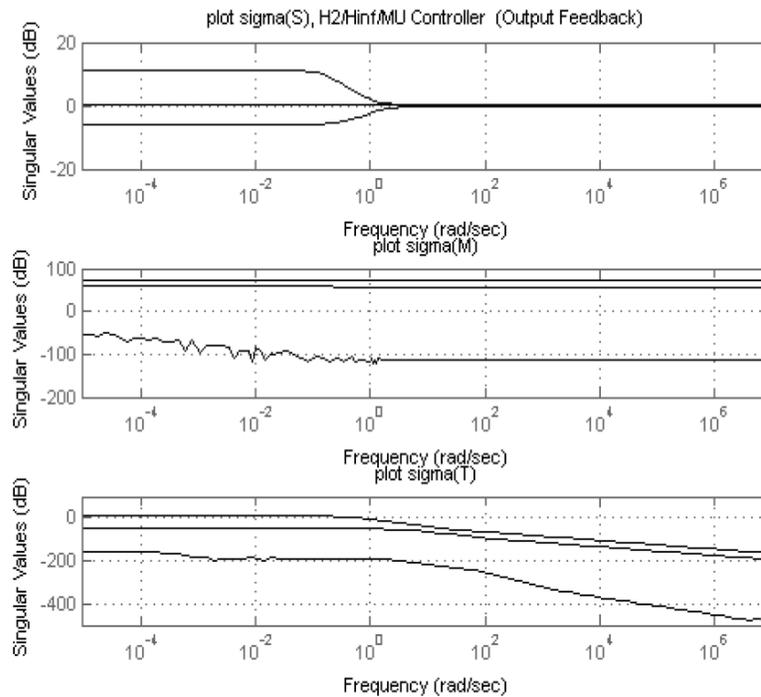


Fig.11. Singular value for T complementary sensitivity function.

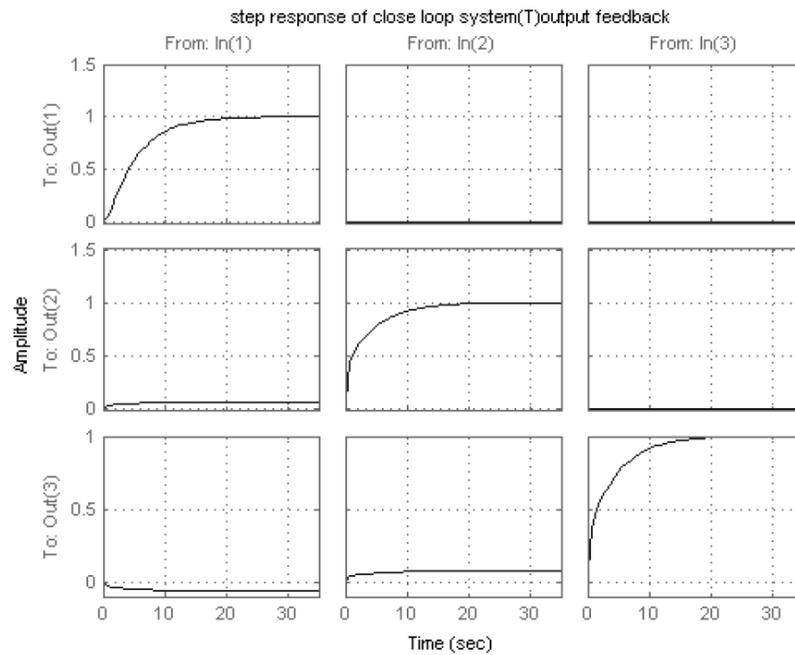


Fig. 12. Step responses for T complementary sensitivity function.

weight and sensitivity functions are the shape of matrix. The sign of success is the combination of nominal and robust performance, together. Accomplishment in reaching to the targets with the minimum controlling signal is of the gains of noted controller. The open-loop system is an unstable one, but the close-loop system showed appropriate results.

## CONCLUSION

The paper brings out a global approach for robust static output-feedback design in which the multiple specifications can be simultaneously defined. In this paper, the problem of the mixed  $H_2/H_\infty/\mu$  robust control was addressed. An algorithm based on LMIs has been proposed in order to find a reach to a static output feedback robust controller which minimizes the cost of an  $H_2$  performance subjected to  $H_\infty$  norm and  $\mu$  synthesis constraints. Each of the nominal and the robust performances has their own strengths and weaknesses. The availability of robust performance has lead to the intensive limitations on the controller, which sometimes exhales it from a possible problem (Keel *et al.*, 1997). Also, availability of the nominal performance means considering the system operation without uncertainty, and it is usual that the essence of uncertainty has a decisive effect on the operation of the system. New approach of this paper is a combination of two controllers of  $\mu$  and  $H_2/H_\infty$  based on output feedback. The controllers for robust stability status, nominal performance, robust performance and noise reduction are continuously designed. First, the controller of  $H_2/H_\infty$  will be designed for nominal performance targets, robust stability and reduction of noise, and then  $\mu$  controller will be designed for robust performance. Afterwards, these two controllers are added up and their weights would be obtained through LMI. Also, the controller will be achieved from solution of the optimization problem. This paper attempts to present an implementation approach for the multivariable controlling systems. In operation, we look for minimization of the faults. If the available error function is not desirable, the use of a suitable weight function can lead us to the target. So, design of the weight function is extremely important. By knowing the data of the problem, we will have the information on which of the frequencies has more uncertainty effect. It is obvious that the controller effect of  $\mu$  should be considered here. Because of multivariable exceptional values system, controller and considered inputs-outputs were shown. Using two low pass filters and one high pass filter for  $H_2/H_\infty$  controller, we tended to optimize the solutions. First, the equations of X-29 aircraft state space are written. Then, the robust static output-feedback controller will be designed. The results which are shown in the figures indicate that the unstable system becomes stable in the presence of uncertainty using the proposed controller. The results are

also indicative that the proposed approach shows an appropriate and desired performance.

## ACKNOWLEDGMENT

The authors wish to thank Prof. A. Khaki-Sedigh for helpful discussions.

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Received: June 19, 2013; Revised: March 17, 2014;  
Accepted: April 4, 2014